DIVISORIAL MODELS OF NORMAL VARIETIES

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ABSTRACT. We prove that the canonical ring of a canonical variety in the sense of [dFH09] is finitely generated. We prove that canonical varieties are klt if and only if $\mathscr{R}(-K_X)$ is finitely generated. We introduce a notion of nefness for non-$\mathbb{Q}$-Gorenstein varieties and study some of its properties. We then focus on these properties for non-$\mathbb{Q}$-Gorenstein toric varieties.

1. INTRODUCTION

In this paper we continue the investigation of singularities of non-$\mathbb{Q}$-Gorenstein varieties initiated in [dFH09] and [Urb11]. In particular we focus on the study of canonical singularities and non-$\mathbb{Q}$-Gorenstein toric varieties.

In the third section we show that if X is canonical in the sense of [dFH09], then the relative canonical ring $\mathscr{R}_X(K_X)$ is a finitely generated $\mathcal{O}_X$-algebra (Theorem 3.4). Thus, if X is canonical, there exists a small proper birational morphism $\pi : X' \to X$ such that $K_{X'}$ is $\mathbb{Q}$-Cartier and π-ample. As a corollary we obtain that the canonical ring of any normal variety with canonical singularities (in the sense of [dFH09]) is finitely generated.

We next turn our attention to log-terminal singularities. Recall that in [Urb11] we gave an example of canonical singularities that are not log-terminal. In this paper we show that, if X is canonical, then finite generation of the relative anti-canonical ring $\mathscr{R}_X(-K_X)$ is equivalent to X being log-terminal (Proposition 3.7).

In the fourth section we introduce a notion of nefness for Weil divisors (on non-$\mathbb{Q}$-factorial varieties). We call such divisors quasi-nef (q-nef) and we study their basic properties. We prove that if X ia a normal variety with canonical singularities such that K_X is q-nef, then $X' = \mathrm{Proj}_X(\mathscr{R}_X(K_X))$ is a minimal model.

In the last two sections, we focus our attention on toric varieties. We give a complete description of quasi-nef divisors on toric varieties and we notice that they correspond to divisors whose divisorial sheaf is globally generated.

In the last section we give a new natural definition of minimal log discrepancies (MLD) in the new setting and we prove that even in the toric case they do not satisfy the ACC conjecture.

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2. BACKGROUND

We work over the complex numbers.

Recall the following definition of de Fernex and Hacon (cf [dFH09]):

Definition 2.1. Let $f : Y \to X$ a proper projective birational morphism of normal varieties. Given any Weil divisor D on X, we define the pullback of D on Y as:

$$f^*(D) = \sum_{P \text{ prime on } Y} \lim_{k\to\infty} \frac{v_P(\mathcal{O}_Y \cdot \mathcal{O}_X(-k!D))}{k!}.$$

Note that if D is $\mathbb{Q}$-Cartier, then $f^*(D)$ coincides with the usual notion of pullback.

Using this definition, de Fernex and Hacon define canonical and log terminal singularities for non-$\mathbb{Q}$-Gorenstein varieties. As usual, this is done in terms of the relative canonical divisor $K_{Y/X}$, for $f : Y \to X$ a proper morphism. Note however that there are two different choices for the relative canonical divisor (which coincide in the $\mathbb{Q}$-Gorenstein setting):

$$K^-_{Y/X} := K_Y - f^*(K_X) \quad \text{and} \quad K^+_{Y/X} := K_Y + f^*(-K_X).$$

We will not use $K^-_{Y/X}$ in this paper, but recall that it is the one used to define log terminal singularities and multiplier ideal sheaves.

Definition 2.2. X is said to be *canonical* if

$$\mathrm{ord}_F(K^+_{Y/X}) > 0$$

for every exceptional prime divisor F over X.

Definition 2.3. X is said to be *log terminal* if and only if there is a an effective $\mathbb{Q}$-divisor Δ such that:

- Δ is a boundary ($K_X + \Delta$ is $\mathbb{Q}$-Cartier) and
- (X, Δ) is klt.

3. Canonical singularities

In this section we will show that if X has canonical singularities, then its canonical ring is finitely generated.

de Fernex and Hacon gave the following characterization of canonical singularities:

Proposition 3.1. [dFH09, Proposition 8.2] *Let X be a normal variety. Then X is canonical if and only if for all sufficiently divisible $m \geq 1$, and for every resolution $f : Y \to X$, there is an inclusion*

$$\mathcal{O}_X(mK_X) \cdot \mathcal{O}_Y \subseteq \mathcal{O}_Y(mK_Y)$$

as sub-$\mathcal{O}_Y$-modules of $\mathcal{K}_Y$.

Lemma 3.2. [Urb11, Lemma 2.14] *Let $f : Y \to X$ be a proper birational morphism such that Y is $\mathbb{Q}$-Gorenstein with canonical singularities. If $\mathcal{O}_Y \cdot \mathcal{O}_X(mK_X) \subseteq \mathcal{O}_Y(mK_Y)$ for any sufficiently divisible $m \geq 1$, then X is canonical.*

The following immediate corollary of Lemma 3.2 is very useful.

Corollary 3.3. *Let $f : Y \to X$ be a proper birational morphism such that Y is $\mathbb{Q}$-Gorenstein and canonical. If $\mathrm{val}_F(K^+_{Y/X}) \geq 0$ for all divisors F on Y, then X is canonical.*

Proof. Recall that $f^\natural(D) := \mathrm{div}(\mathcal{O}_X(-D) \cdot \mathcal{O}_Y)$.

For all sufficiently divisible $m \geq 1$, $\mathrm{val}_F(K^+_{m,Y/X}) \geq 0$ (i.e. $mK_Y \geq -f^\natural(-mK_X)$), so that:

$$\mathcal{O}_Y \cdot \mathcal{O}_X(mK_X) \hookrightarrow (\mathcal{O}_Y \cdot \mathcal{O}_X(mK_X))^{\vee\vee} = \mathcal{O}_Y(-f^\natural(-mK_X)) \hookrightarrow \mathcal{O}_Y(mK_Y).$$

□

The first result that we will prove is that if X is canonical, then $\mathscr{R}_X(K_X)$ is finitely generated over X. Note that this result is trivial for $\mathbb{Q}$-Gorenstein varieties.

Theorem 3.4. *If X is canonical, then $\mathscr{R}_X(K_X)$ is finitely generated over X.*

Proof. We may assume that X is affine. Let $\tilde{X} \to X$ be a resolution. By [BCHM06] $\mathscr{R}(K_{\tilde{X}}/X)$ is finitely generated. Running the MMP over X, we obtain $X^c = \mathrm{Proj}_X(\mathscr{R}(K_{\tilde{X}}))$ and let $f : X^c \to X$ be the induced morphism, where X^c is canonical. Since X is canonical, for any $m > 0$, there is an inclusion $\mathcal{O}_{X^c} \cdot \mathcal{O}_X(mK_X) \to \mathcal{O}_{X^c}(mK_{X^c})$. Pushing this forward we obtain inclusions

$$f_*(\mathcal{O}_{X^c} \cdot \mathcal{O}_X(mK_X)) \subset f_*\mathcal{O}_{X^c}(mK_{X^c}) \subset \mathcal{O}_X(mK_X).$$

Since the left and right hand sides have isomorphic global sections, then $H^0(f_*\mathcal{O}_{X^c}(mK_{X^c})) \cong H^0(\mathcal{O}_X(mK_X))$. Since X is affine, $\mathcal{O}_X(mK_X)$ is globally generated and hence $f_*\mathcal{O}_{X^c}(mK_{X^c}) = \mathcal{O}_X(mK_X)$. But then $\mathscr{R}(K_X/X) \cong \mathscr{R}(K_{X^c}/X)$ is finitely generated. □

Remark 3.5. Note that we have seen that

$$\mathscr{R}(K_X/X) \cong \mathscr{R}(K_{X^c}/X) \cong \mathscr{R}(K_{\tilde{X}}/X)$$

hence

$$X^c = \mathrm{Proj}_X(\mathscr{R}(K_X/X))$$

and so $X^c \to X$ is a small mophism.

Corollary 3.6. *If X is canonical, then the canonical ring $\mathscr{R}(K_X)$ is finitely generated.*

Proof. Since $f : X^c \to X$ is small, it follows that $\mathscr{R}(K_X) \cong \mathscr{R}(K_{X^c})$. Since X^c is canonical and $\mathbb{Q}$-Gorenstein if follows that $\mathscr{R}(K_{X^c})$ is finitely generated (cf. [BCHM06]).

□

The next Proposition, srtictly relates log terminal singulatities with the finite generation of the canonical ring even in the non-$\mathbb{Q}$-Gorenstein case:

Proposition 3.7. *Let X be a normal variety with at most canonical singularities. $\mathscr{R}(-K_X/X)$ is a finitely generated $\mathcal{O}_X$-algebra if and only if X is log terminal.*

Proof. If X is log terminal, then $\mathscr{R}(-K_X/X)$ is a finitely generated $\mathcal{O}_X$-algebra by [Kol08, Theorem 92].

For the reverse implication, since $\mathscr{R}(-K_X/X)$ is finitely generated, by [KM98, Proposition 6.2], there exists a small map $\pi : X^- \to X$, such that $-K_{X^-} = \pi_*^{-1}(-K_X)$ is $\mathbb{Q}$-Cartier and π-ample. For any m sufficiently divisible, consider the natural map $\mathcal{O}_{X^-} \cdot \mathcal{O}_X(-mK_X) \to \mathcal{O}_{X^-}(-mK_{X^-})$. Since $-K_{X^-}$ is π-ample, we can choose $A \subseteq X$ an ample divisor so that $-K_X + A$ and $-K_{X^-} + g^*A$ are both globally generated. Since the small map induces an isomorphism at the level of global sections and the two sheaves are globally generated,

$$\mathcal{O}_{X^-} \cdot \mathcal{O}_X(-mK_X + A) \to \mathcal{O}_{X^-}(-mK_{X^-} + g^*A)$$

is an isomorphism of sheaves. Thus, considering $f : Y \to X$ and $g : Y \to X^-$, a common log resolution of both X and X^-, we have

$$K_Y + \frac{1}{m}g^*(-mK_{X^-}) = K_Y + \frac{1}{m}g^*(\pi^\natural(-mK_X)) = K_Y + \frac{1}{m}f^\natural(-mK_X) \geq 0$$

so that X^- has at most canonical singularities. Since K_{X^-} is $\mathbb{Q}$-Cartier and canonical, X^- is log terminal.
Choosing a general ample $\mathbb{Q}$-divisor $H^- \sim_{\mathbb{Q},X} -K_{X^-}$, let $m \gg 0$ and $G^- \in |mH^-|$ a general irreducible component. Picking $\Delta^- := \frac{G^-}{m}$ then $K_{X^-} + \Delta^- \sim_{\mathbb{Q},X} 0$ is still log terminal and hence so is $(X, \Delta = \pi_*\Delta^-)$. $\square$

4. Quasi-nef divisors

Given a divisor D on a variety X, it is useful to know if the divisor is nef. In particular, varieties such that the canonical divisor K_X is nef, are minimal models.
For arbitrary normal varieties, unfortunately, there is no good notion of nefness (this is a numerical property that is well defined if the variety is $\mathbb{Q}$-factorial). In particular, whenever looking for a minimal model in this case it is always necessary to either pass to a resolution of the singularities or to perturb the canonical divisor adding a boundary (an auxiliary divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier). However both operations are not canonical and in either cases different choices lead us to different minimal models. What we would like to do in this section is to define a notion of a minimal model for an arbitrary normal variety.

We will start defining a notion of nefness for a divisor that is not $\mathbb{Q}$-Cartier.

Definition 4.1. Let X be a normal variety. A divisor $D \subseteq X$ is *quasi-nef* (*q-nef*) if for every ample $\mathbb{Q}$-divisor $A \subseteq X$, $\mathcal{O}_X(m(D + A))$ is generated by global sections for every $m > 0$ sufficiently divisible.

Remark 4.2. Let X be a normal $\mathbb{Q}$-factorial variety. A divisor $D \subseteq X$ is nef if and only if it is q-nef.

Proposition 4.3. *Let D be a divisor on a normal variety X. If $g : Y \to X$ is a small projective birational map such that $\bar{D} := g_*^{-1}D$ is $\mathbb{Q}$-Cartier and g-ample, then D is q-nef if and only if $\bar{D}$ is nef.*

Proof. Let us first assume that D is q-nef. For every ample divisor $A \subseteq X$, by definition there exists a positive integer m such that $\mathcal{O}_X(m(D+A))$ is generated by global sections and $\mathcal{O}_Y(m\bar{D})$ is relatively globally generated. In particular, since g is small,

$$\varphi : \mathcal{O}_Y \cdot \mathcal{O}_X(m(D+A)) \to \mathcal{O}_Y(m(\bar{D} + g^*A))$$

induces an isomorphism at the level of global sections. Now $\bar{D}$ is g-ample, and there exists $k \gg 0$ such that $\mathcal{O}_Y(m(\bar{D}+g^*A)) \otimes \mathcal{O}_Y(kg^*A)$ is also generated by global sections, hence φ must be surjective and hence an isomorphism. Since $\mathcal{O}_Y \cdot \mathcal{O}_X(m(D+A))$ is generated by global sections, so is $\mathcal{O}_Y(m(\bar{D}+g^*A))$. This implies that $\bar{D}+g^*A$ is nef, and since nefness is a closed property, $\bar{D}$ is nef.

Let us now suppose that $\bar{D}$ is a nef divisor on Y. Fix an ample divisor A on X and r an integer such that $rA \sim H$ is very ample. Since $\bar{D}$ is g-ample, $\bar{D} + kg^*(A)$ is an ample divisor for any k big enough. Fix k with this property. In particular, since $\bar{D}$ is nef, by Fujita's vanishing theorem ([Laz04a, Theorem 1.4.35]) we have that

$$H^i(Y, k(m-(n-i))(\bar{D}+g^*(A))) = H^i(Y, (m-(n-i))(\bar{D}+kg^*(A))+(m-(n-i))(k-1)\bar{D}) = 0$$

for $0 < i \leq n = \dim X$, if $m \gg 0$. By [Laz04a, Lemma 4.3.10], this implies that

$$R^j g_* \mathcal{O}_Y(mk(\bar{D}+g^*A)) = 0 \quad \text{for } j > 0 \tag{1}$$

and even more, via the projection formula, we have

$$R^j g_* \mathcal{O}_Y(mk(\bar{D}+g^*A) - (n-i)g^*A) = 0 \quad \text{for } j > 0,\ 0 \leq i \leq n. \tag{2}$$

In particular, with a spectral sequence computation we have that all the cohomologies of the above sheaves vanish. Hence by Castelnuovo-Mumford regularity we conclude that $f_*(\mathcal{O}_Y(mk(\bar{D}+g^*A)))$ is generated by global sections. Let us denote $\mathcal{F} := \mathcal{O}_Y(mk(\bar{D}+g^*A)+nr(\bar{D}+g^*A))$, and $M := (mk+nr)$.
We now consider the following exact sequence:

$$0 \to g_*\mathcal{F} \to \mathcal{O}_X(M(D+A)) \to Q \to 0$$

where Q is the cokernel of the first map. We will prove by induction on $d := \dim(\mathrm{Supp}(Q))$, that $\mathcal{O}_X(M(D+A))$ is generated by global sections. If $\dim(\mathrm{Supp}(Q)) = 0$, then Q is supported on points and hence globally generated. Since $g_*\mathcal{F}$ is globally generated as we observed above, and $H^1(g_*\mathcal{F}) = 0$, it follows that $\mathcal{O}_X(M(D+A))$ is globally generated.
Let us now consider the general case, with $\dim Q = d$. In particular, if $H_1, \ldots, H_d \in |rA|$ are general hyperplane sections, $g_*\mathcal{F}|_{H_1 \cap \cdots \cap H_d}$ is torsion free and we can construct the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & g_*\mathcal{F} \arrow[r] \arrow[d, "p"] & \mathcal{O}_X(M(D+A)) \arrow[r] \arrow[d, "q"] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & g_*\mathcal{F}|_{H_1\cap\cdots\cap H_d} \arrow[r, "s"] \arrow[d] & \mathcal{O}_X(M(D+A))|_{H_1\cap\cdots\cap H_d} \arrow[r, "v"] \arrow[d] & Q|_{H_1\cap\cdots\cap H_d} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We first need to justify the existence of the map s. It suffices show that $(g_*\mathcal{F}|_{H_1\cap\cdots\cap H_d})^{\vee\vee} \cong \mathcal{O}_X(M(D+A))|_{H_1\cap\cdots\cap H_d}$, where we already know that the two sheaves agree on a big open set. $\mathcal{O}_X(M(D+A))$ is a reflexive sheaf if and only if there exists an associated exact sequence of the form

$$0 \to \mathcal{O}_X(M(D+A)) \to \mathscr{E} \to \mathscr{G} \to 0$$

where $\mathscr{E}$ is locally free and $\mathscr{G}$ is torsion free [Har80, Proposition 1.1]. Moreover, we need to show that the restriction to a general hyperplane section H leaves the sequence exact. In particular we need to show that $\mathscr{G}|_H$ is torsion free, and this is true since it is possible to pick H that does not contain any of the associated primes of $\mathscr{G}$. Since the left hand side is reflexive by definition, and the two sheaves agree on a big open set, they have to be the same.

We can now finish the proof via a few simple observations. In fact, we have that $\dim \operatorname{Supp}(Q|_{H_1\cap\cdots\cap H_d}) = 0$, hence this sheaf is generated by global sections. Also, the map s is an isomorphism at the level of global sections, since the original map g is small and the hyperplanes are general, and by (2) it follows that $H^1(g_*\mathcal{F}|_{H_1\cap\cdots\cap H_d}) = 0$. Hence $Q|_{H_1\cap\cdots\cap H_d}$ is trivial so that Q is trivial itself and $\mathcal{O}_X(M(D+A)) \cong g_*\mathcal{F}$ is generated by global sections.

We conclude that for every ample divisor A on X, there exist an integer M such that $\mathcal{O}_X(M(D+A))$ is generated by global sections, hence D is q-nef. $\square$

Definition 4.4. Let X be a normal projective variety, D any divisor on X and A an ample divisor. If there exists a $t \in \mathbb{R}$ such that $D + tA$ is quasi-nef, we define the quasi-nef threshold with respect to A (qnt_A) as:

$$\mathrm{qnt}_A(D) = \inf\{t \in \mathbb{R} | \mathcal{O}_X(m(D+tA)) \text{ is globally generated for all } m \text{ sufficiently divisible}\}.$$

Remark 4.5. Let X be a normal projective variety with at most log-terminal singularities. For any divisor D on X and any ample divisor A, then $\mathrm{qnt}_A(D)$ exists and it is a rational number. This is a direct consequence of the fact that for any variety with at most klt singularities, every divisorial ring is finitely generated [Kol08, Theorem 92].

Let us recall the following conjecture from [Urb11]:

Conjecture 4.6. *Let X be a projective normal variety. Then, for any divisor $D \in \mathrm{WDiv}_{\mathbb{Q}}(X)$, there exists a very ample divisor A such that $\mathcal{O}_X(mD) \otimes \mathcal{O}_X(A)^{\otimes m}$ is globally generated for every $m \geq 1$.*

5. Quasi-nef Divisors on Toric Varieties

For the notation and basic properties of toric varieties we refer the reader to [CLS11].

Consider a normal projective toric variety $X = X_\Sigma$ corresponding to a complete fan Σ in $N_\mathbb{R}$ (with no torus factor), with $\dim N_\mathbb{R} = n$. Recall that every T_N-invariant Weil divisor is represented by a sum

$$D = \sum_{\rho \in \Sigma(1)} d_\rho D_\rho,$$

where ρ is a one-dimensional subcone (a ray), and D_ρ is the associated T_N-invariant prime divisor. D is Cartier if for every maximal dimension subcone $\sigma \in \Sigma(n)$, $D|_{U_\sigma}$ is locally a divisor of a character $\mathrm{div}(\chi^{m_\sigma})$, with $m_\sigma \in N^\vee = M$. If D is Cartier we will say that $\{m_\sigma | \sigma \in \Sigma(n)\}$ is the Cartier data of D.

To every divisor we can associate a polyhedron:

$$P_D = \{m \in M_\mathbb{R} | \langle m, u_\rho \rangle \geq -d_\rho \text{ for every } \rho \in \Sigma(1)\}.$$

Even if the divisor is not Cartier, the polyhedron is still convex and rational but not necessarily integral.

For every divisor D and every cone $\sigma \in \Sigma(n)$, we can describe the local sections as

$$\mathcal{O}_X(D)(U_\sigma) = \mathbb{C}[W]$$

where $W = \{\chi^m | \langle m, u_\rho \rangle + d_\rho \geq 0 \text{ for all } \rho \in \sigma(1)\}$.

Let us recall the following Proposition from [Lin03]:

Proposition 5.1. *For a torus invariant Weil divisor $D = \sum d_\rho D_\rho$, the following statements hold.*

(1) $\Gamma(X, D) = \bigoplus_{m \in P_D \cap M} \mathbb{C} \cdot \chi^m$.

(2) Given that $\mathcal{O}_X(D)(U_\sigma) = \mathbb{C}[\sigma^\vee \cap M]\langle \chi^{m_{\sigma,1}}, \dots, \chi^{m_{\sigma,r_\sigma}} \rangle$ is a finitely generated $\mathbb{C}[\sigma^\vee \cap M]$ module for every $\sigma \in \Sigma(n)$ and a minimal set of generators is assumed to be chosen, $\mathcal{O}_X(D)$ is generated by its global sections if and only if $m_{\sigma,j} \in P_D$ for all σ and j.

We will also need the following result [Eli97].

Theorem 5.2. *Let X be a complete toric variety and let D be a Cartier divisor on X. Then the ring*

$$\mathscr{R}_D := \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD))$$

is a finitely generated $\mathbb{C}$-algebra.

Corollary 5.3. *Since every toric variety admits a $\mathbb{Q}$-factorialization, a small morphism from a $\mathbb{Q}$-factorial variety (*[Fuj01, Corollary 3.6]*), the above result holds for Weil divisors as well.*

Remark 5.4. Conjecture 4.6 holds for $X = X_\Sigma$, a complete toric variety.

We can now focus our attention on q-nef divisors.

Remark 5.5. A small birational map $f : Y \to X$ is given by adding faces of dimension ≥ 2 to the fan. This operation increases the number of subcones. In particular a subcone in the fan corresponding to Y may be strictly contained in one of the original subcones.

Remark 5.6. Let us consider a Weil divisor $D \subseteq X$ on a normal toric variety. If $f : Y \to X$ is a small birational map, then $P_D = P_{f_*^{-1}D}$. This is clear, since the definition of the polyhedron only depends on the rays generating the fan and not on the structure of the subcones.

We assume that the polyhedron P_D is of maximal dimension and that zero is inside the polyhedron.

To have a better description of the relation between a small morphism and the local sections of a Weil divisor, we will introduce a new polyhedron associated to the divisor, the dual of P_D.

Definition 5.7. Let $D = \sum d_\rho D_\rho \subseteq X = X_\Sigma$ be a Weil divisor on a normal toric variety. We define $Q_D \subseteq N_\mathbb{R}$ to be the convex hull generated by $\frac{1}{d_\rho} u_\rho$ where $\rho \in \Sigma(1)$. In Particular

$$Q_D = P_D^* = \{u \in N_\mathbb{R} | \langle m, u \rangle \geq -1 \text{ for all } m \in P_D\}.$$

Recall that a divisor D is Cartier if and only if for each $\sigma \in \Sigma$, there is $m_\sigma \in M$ with $\langle m_\sigma, u_\rho \rangle = -d_\rho$ for all $\rho \in \sigma(1)$, with $D|_{U_\sigma} = \operatorname{div}(\chi^{-m_\sigma})$ ([CLS11, Theorem 4.2.8]).

We will define Σ' to be the fan generated by Σ and the faces of Q_D. In particular, any face of Q_D is contained in a hyperplane corresponding to m_σ for some $\sigma \in \Sigma(n)$. Note that the vertices of Q_D are all contained in the 1-dimensional faces of Σ, hence $Y := X_{\Sigma'} \to X$ is a small birational map.

For every cone $\sigma \in \Sigma(n)$, if $\mathcal{O}_X(D)(U_\sigma)$ is locally generated by a single equation (is locally Cartier) nothing changes. Otherwise we substitute the cone σ by the set of subcones generated by the faces of Q_D contained in σ.

Lemma 5.8. *With the notation above, suppose that $\bar{\sigma} \subseteq \Sigma'(n)$ corresponds to a face of Q_D. Let $\bar{m} \in M_\mathbb{R}$ the element corresponding to the hyperplane containing the face, hence $\mathcal{O}_X(D)(U_{\bar{\sigma}}) = \mathbb{C}[\bar{\sigma}^\vee \cap M]\langle \chi^{\bar{m}} \rangle$. If $\mathcal{O}_X(D)$ is globally generated, then $\bar{m} \in P_D$.*

Proof. Since $\chi^{\bar{m}}$ is a generator of $\mathcal{O}_X(D)(U_{\bar{\sigma}})$, we have that $\langle \bar{m}, u_\rho \rangle = -d_\rho$ for every $\rho \in \bar{\sigma}(1)$. Also, since Q_D is convex and $\langle \bar{m}, 0 \rangle = 0 > -1$, we have that $\langle \bar{m}, u_\rho \rangle \geq -d_\rho$ for every $\rho \in \Sigma(1)$, hence $\bar{m} \in P_D$. □

Proposition 5.9. *Let X be a normal toric variety and D a Weil divisor whose corresponding reflexive sheaf is generated by global sections. Then there exists a small map $f : Y \to X$ of toric varieties such that $\bar{D} := f_*^{-1}D$ is $\mathbb{Q}$-Cartier and f-ample, and the vertices of P_D are given by the Cartier data $\{m_\sigma | \sigma \in \Sigma'(n)\}$ of $\bar{D}$, where Σ' is the fan associated to Y.*

Proof. We will consider the toric variety associated to the fan Σ' generated by Σ and the convex polytope Q_D. It follows from the construction that the divisor $\bar{D}$ is $\mathbb{Q}$-Cartier. By

Lemma 5.8, since Q_D is convex, we have that the reflexive sheaf corresponding to $\bar{D}$ is still generated by global sections.

Even more, every curve C extracted via the map f will correspond to a face $\tau \subseteq \Sigma'$. Since $\bar{D}$ is globally generated, we already know that $(\bar{D}.C) \geq 0$. In particular τ is given as the intersection of two maximal cones $\tau = \sigma \cap \sigma'$, and for each of the cones we have local generators of $\bar{D}$, m and m'. The intersection is computed as $(\bar{D}.C) = \langle m, u \rangle - \langle m', u \rangle$, where u is a ray in $\sigma \backslash \sigma'$, where this is zero if and only if $m = m'$, and this would not be one of the curves to be extracted by the map f by definition.

□

Remark 5.10. Because of Proposition 5.9 it makes sense to define a Weil divisor D on a normal toric variety X to be q-nef if $\mathcal{O}_X(mD)$ is globally generated for $m \gg 0$.

Example 5.11. Let X be a normal projective toric variety and $A = \sum a_\rho D_\rho \subseteq X$ an ample divisor. Then $\mathrm{qnt}_A(D)$ can be explicitely computed. In particular, let $D = \sum d_\rho D_\rho$ any Weil divisor. If no multiple of D is globally generated, this implies that for every $b \in \mathbb{N}$, there exists $\sigma_b \in \Sigma(n)$ such that $u_{\rho_b} \notin \sigma_b$ and $\langle m^{bD}_{\sigma_b}, u_{\rho_b} \rangle < -bd_{\rho_b}$, where $m^{bD}_{\sigma_b}$ is one of the generators of $\mathcal{O}_X(bD)(U_{\sigma_b})$, i.e. there exists a positive rational number δ_{ρ_b} such that $\langle m^{bD}_{\sigma_b}, u_{\rho_b} \rangle = -bd_{\rho_b} - \delta_{\rho_b}$. Since A is ample, the support function of A is strictly convex, and in particular $\langle m^{A}_{\sigma_a}, u_{\rho_b} \rangle > -a_{\rho_b}$, i.e. there exists a positive rational number ε_{ρ_b} such that $\langle m^{A}_{\sigma_b}, u_{\rho_b} \rangle = -a_{\rho_b} + \varepsilon_{\rho_b}$.

For every $\rho_b \notin \sigma_b$ and every σ_b we can find a rational number t_{b,σ_b,ρ_b} so that:

$$\langle m^{D}_{\sigma_b} + t_{b,\sigma_b,\rho_b} m^{A}_{\sigma_b}, u_{\rho_b} \rangle = -bd_{\rho_b} - \delta_{\rho_b} - t_{b,\sigma_b,\rho_b} a_{\rho_b} + t_{b,\sigma_b,\rho_b} \varepsilon_{\rho_b} = -bd_{\rho_b} - t_{b,\sigma_b,\rho_b} a_{\rho_b}.$$

Then

$$\mathrm{qnt}_A(D) = \inf_b \max_{\sigma_b, \rho_b \notin \sigma_b} -t_{b,\sigma_b,\rho_b}.$$

6. Minimal Log Discrepancies for Terminal Toric Threefolds

In this last section we go back to properties of log discrepancies in the setting of [dFH09] in the context of toric varieties.

Depending on our choice of a relative canonical divisor, we have two possible definitions for the Minimal Log Discrepancies (MLD's).

Definition 6.1. Let X be a normal variety over the complex numbers, we associate two numbers to the variety X:

$$\mathrm{MLD}^{-}(X) = \inf_E \mathrm{val}_E(K^{-}_{Y/X})$$

and

$$\mathrm{MLD}^{+}(X) = \inf_E \mathrm{val}_E(K^{+}_{Y/X})$$

where $E \subseteq Y$ is any prime divisor and $Y \to X$ is any proper birational morphism of normal varieties.

It is natural to wonder if these MLD's also satisfy the ACC conjecture. If X is assumed to be $\mathbb{Q}$-Gorenstein, then this is conjectured to hold by V. Shokurov. In view of [dFH09, Theorem 5.4], the MLD$^+$'s correspond to MLD's of appropriate pairs (X, Δ). However the coefficients of Δ do not necessarily belong to a DCC set (cf. [Amb06]).

Proposition 6.2. *The set of all MLD$^+$'s for terminal toric threefolds does not satisfy the ACC conjecture.*

Proof. We give an explicit example of a set of terminal toric threefolds whose associeted MLD$^+$'s converge to a number from below. The problem is local, hence we will consider a set of affine toric threefolds given by the following data.

Let X be the affine toric variety associated to the cone $\sigma = \langle u_1, u_2, u_3, u_4 \rangle$, $u_1 = (2, -1, 0)$, $u_2 = (2, 0, 1)$, $u_3 = (1, 1, 1)$, $u_4 = (a, 1, 0)$ with $a \in \mathbb{N}$. The associated toric variety in non-$\mathbb{Q}$-Gorenstein, i.e. the canonical divisor $K_X = \sum -D_i$ is not $\mathbb{Q}$-Cartier.

Let $\Delta = \sum d_i D_i$ be a $\mathbb{Q}$-divisor such that $0 \leq d_i \leq 1$ and $-K_X + \Delta$ is $\mathbb{Q}$-Cartier. This means that there exists $m = (x, y, z)$ such that $-K_X + \Delta = \sum (m, u_i) D_i$. Hence $\Delta = (2x - y - 1)D_1 + (2x + z - 1)D_2 + (x + y + z - 1)D_3 + (ax + y - 1)D_4$.

The exceptional divisor E giving the smallest discrepancy is the one corresponding to the element $u_E = u_1 + u_2 + u_3$ (it is the exceptional divisor generated by the ray of smallest norm). In particular, we have

$$\mathrm{val}_E(K^+_{Y/X}) = \inf_{\Delta \text{ boundary}} 5x - 2z.$$

Increasing the value of the parameter a we see that the minimal valuation (solving a problem of minimality with constrains) is given by $\dfrac{4a+5}{a+2}$ which accumulates from below at the value 4. $\square$

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